

# Hot and dense gas of quark quasi-particles

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Some features of hot and dense gas of quarks which are considered as the quasi-particles of the model Hamiltonian with four-fermion interaction are studied. Being adapted to the Nambu-Jona-Lasinio model this approach allows us to accommodate a phase transition similar to the nuclear liquid-gas one at the proper scale. It allows us to argue an existence of the mixed phase of vacuum and baryonic matter (even at zero temperature) as a plausible scenario of chiral symmetry (partial) restoration.

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Understanding in full and describing dependably the critical phenomena (chiral and deconfinement phase transitions) in QCD is still elusive because of a necessity to have the corresponding efficient non-perturbative methods for strongly coupled regime to be analyzed. For the time being such studies are pursued by invoking diverse effective models. The Nambu-Jona-Lasinio(NJL)-type models are certainly playing the most advanced role in this analysis [1]. This approach deals with the four-fermion interactions in lieu of a gluon field QCD dynamics and does not incorporate (albeit being supplemented by the Polyakov loops it does [2]) the property of confinement. In the meantime it is quite successful in realizing the spontaneous breakdown of chiral symmetry and its restoration at non-zero temperatures or quark densities. Apparently, the non-renormalizable nature of the NJL model supposes another approximations introduced to be solved and might lead to some conclusions which are dependent on the regularization scenario sometimes. Hence, it requires a steadfast control of all inputs done to have the consistent physical results and to avoid the reasons for skeptical attitude.

Instructive example was given in Refs. [3] and [4] in which the ground state of model Hamiltonian with four-fermion interactions was studied in detail. The quarks were treated as the quasi-particles of this Hamiltonian and unexpected singularity (discontinuity) of the mean energy functional as a function of the current quark mass was found out. In particular case of the NJL model new solution branches of the equation for dynamical quark mass as a function of chemical potential (the details are shown below in Fig. 1) have been found out, and the appearance of state filled up with quarks which is almost degenerate with the vacuum state both in quasi-particle chemical potential and in ensemble pressure has been discovered.

Here we are going to study the quark ensemble features at finite temperature and fixed baryonic chemical potential and to analyse the first order phase transition which takes place in such a system of free quasi-particles. Analysis is performed within the framework of two approaches which are supplementary, in a sense, albeit give the identical results. One of those approaches, based on the Bogolyubov transformations, is especially informative to study the process of filling the Fermi sphere up because at this point the density of quark ensemble develops a continuous dependence on the Fermi momentum. It allows us to reveal an additional structure in the solution of the gap equation for dynamical quark mass just in the proper interval of parameters characteristic for phase transition and to trace its evolution. The result is that quark ensemble might be found in two aggregate states, gas and liquid, and the chiral condensate is partially restored in a liquid phase. In order to make these conclusions easily perceptible we deal with the simplest version of the NJL model (with one flavor and one of the standard parameter sets) and, actually, are not targeted to adjust the result obtained with well-known nuclear liquid-gas phase transition. Besides, it seems our approach might be treated as a sort of microscopic ground of the conventional bag model and those states filled up with quarks are conceivable as a natural 'construction material' for baryons.

Now as an input to start with we remind the key moments of approach developed. The corresponding model Hamiltonian includes the interaction term taken in the form of a product of two colored currents located in the spatial points  $\mathbf{x}$  and  $\mathbf{y}$  which are connected by a formfactor and its density reads as

$$\mathcal{H} = -\bar{q}(i\gamma\nabla + im)q - \bar{q}t^a\gamma_\mu q \int d\mathbf{y} \bar{q}'t^b\gamma_\nu q' \langle A_\mu^a A_\nu^b \rangle, \quad (1)$$

where  $q = q(\mathbf{x})$ ,  $\bar{q} = \bar{q}(\mathbf{x})$ ,  $q' = q(\mathbf{y})$ ,  $\bar{q}' = \bar{q}(\mathbf{y})$  are the

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quark and anti-quark operators,

$$q_{\alpha i}(\mathbf{x}) = \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{1}{(2|p_4|)^{1/2}} [a(\mathbf{p}, s, c) u_{\alpha i}(\mathbf{p}, s, c) e^{i\mathbf{p}\mathbf{x}} + b^+(\mathbf{p}, s, c) v_{\alpha i}(\mathbf{p}, s, c) e^{-i\mathbf{p}\mathbf{x}}], \quad (2)$$

$p_4^2 = -\mathbf{p}^2 - m^2$ ,  $i$ -is the colour index,  $\alpha$  is the spinor index in the coordinate space,  $a^+$ ,  $a$  and  $b^+$ ,  $b$  are the creation and annihilation operators of quarks and anti-quarks,  $a|0\rangle = 0$ ,  $b|0\rangle = 0$ ,  $|0\rangle$  is the vacuum state of free Hamiltonian and  $m$  is a current quark mass. The summation over indices  $s$  and  $c$  is meant everywhere, the index  $s$  describes two spin polarizations of quark and the index  $c$  plays the similar role for a colour. As usual  $t^a = \lambda^a/2$  are the generators of  $SU(N_c)$  colour gauge group and  $m$  is the current quark mass. The Hamiltonian density is considered in the Euclidean space and  $\gamma_\mu$  denote the Hermitian Dirac matrices,  $\mu, \nu = 1, 2, 3, 4$ .  $\langle A_\mu^a A_\nu^b \rangle$  stands for the formfactor of the following form

$$\langle A_\mu^a A_\nu^b \rangle = \delta^{ab} \frac{2\tilde{G}}{N_c^2 - 1} [I(\mathbf{x} - \mathbf{y}) \delta_{\mu\nu} - J_{\mu\nu}(\mathbf{x} - \mathbf{y})], \quad (3)$$

where the second term is spanned by the relative distance vector and the gluon field primed denotes that in the spatial point  $\mathbf{y}$ . The effective Hamiltonian density (1) results from averaging the ensemble of quarks influenced by intensive stochastic gluon field  $A_\mu^a$ , see Ref. [3]. For the sake of simplicity we neglect the contribution of the second term in (3) in what follows. The ground state of the system is searched as the Bogolyubov trial function composed by the quark-anti-quark pairs with opposite momenta and with vacuum quantum numbers, i.e.

$$|\sigma\rangle = \mathcal{T} |0\rangle, \quad \mathcal{T} = \Pi_{p,s} \exp\{\varphi[a^+(\mathbf{p}, s)b^+(-\mathbf{p}, s) + a(\mathbf{p}, s)b(-\mathbf{p}, s)]\}. \quad (4)$$

In this formula and below, in order to simplify the notations, we refer to only one complex index which means both the spin and colour polarizations. The parameter  $\varphi(\mathbf{p})$  which describes the pairing strength is determined by the minimum of mean energy

$$E = \langle \sigma | H | \sigma \rangle. \quad (5)$$

By introducing the 'dressing transformation' we define the creation and annihilation operators of quasi-particles as  $A = \mathcal{T} a \mathcal{T}^{-1}$ ,  $B^+ = \mathcal{T} b^+ \mathcal{T}^{-1}$  and for fermions  $\mathcal{T}^{-1} = \mathcal{T}^\dagger$ . Then the quark field operators are presented as

$$q(\mathbf{x}) = \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{1}{(2|p_4|)^{1/2}} [A(\mathbf{p}, s) U(\mathbf{p}, s) e^{i\mathbf{p}\mathbf{x}} + B^+(\mathbf{p}, s) V(\mathbf{p}, s) e^{-i\mathbf{p}\mathbf{x}}], \quad \bar{q}(\mathbf{x}) = \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{1}{(2|p_4|)^{1/2}} [A^+(\mathbf{p}, s) \bar{U}(\mathbf{p}, s) e^{-i\mathbf{p}\mathbf{x}} + B(\mathbf{p}, s) \bar{V}(\mathbf{p}, s) e^{i\mathbf{p}\mathbf{x}}],$$

moreover, the transformed spinors  $U$  and  $V$  are given by the following forms

$$U(\mathbf{p}, s) = \cos(\varphi) u(\mathbf{p}, s) - \sin(\varphi) v(-\mathbf{p}, s), \quad V(\mathbf{p}, s) = \sin(\varphi) u(-\mathbf{p}, s) + \cos(\varphi) v(\mathbf{p}, s). \quad (6)$$

where  $\bar{U}(\mathbf{p}, s) = U^+(\mathbf{p}, s) \gamma_4$ ,  $\bar{V}(\mathbf{p}, s) = V^+(\mathbf{p}, s) \gamma_4$  are the Dirac conjugated spinors.

In the paper Ref. [4] the process of filling in the Fermi sphere with the quasi-particles of quarks was studied by constructing the state of the Sletter determinant type

$$|N\rangle = \prod_{|\mathbf{P}| < P_F; S} A^+(\mathbf{P}; S) |\sigma\rangle, \quad (7)$$

which possesses the minimal mean energy over the state  $|N\rangle$ . The polarization indices run over all permissible values here and the quark momenta are bounded by the limiting Fermi momentum  $P_F$ . The momenta and polarizations of states forming the quasi-particle gas are marked by the capital letters similar to above formula. In all other cases the small letters are used.

As known the ensemble state at finite temperature  $T$  is described by the equilibrium statistical operator  $\rho$ . Here we use the Bogolyubov-Hartree-Fock approximation in which the corresponding statistical operator is presented by the following form

$$\rho = \frac{e^{-\beta \hat{H}_{\text{app}}}}{Z_0}, \quad Z_0 = \text{Tr} \{e^{-\beta \hat{H}_{\text{app}}}\}, \quad (8)$$

where some approximating effective Hamiltonian  $H_{\text{app}}$  is quadratic in the creation and annihilation operators of quark and anti-quark quasi-particles  $A^+$ ,  $A$ ,  $B^+$ ,  $B$  and is defined in the corresponding Fock space with the vacuum state  $|\sigma\rangle$  and  $\beta = T^{-1}$ . We don't need to know the exact form of this operator henceforth because all the quantities of our interest in the Bogolyubov-Hartree-Fock approximation are expressed by the corresponding averages (a density matrix)

$$n(P) = \text{Tr}\{\rho A^+(\mathbf{P}; S) A(\mathbf{P}; S)\},$$

$$\bar{n}(Q) = \text{Tr}\{\rho B^+(\mathbf{Q}; T) B(\mathbf{Q}; T)\},$$

which are found by solving the following variational problem. One needs to determine the statistical operator  $\rho$  in such a form in order to have at the fixed mean charge

$$\bar{Q}_4 = \text{Tr}\{\rho Q_4\} = V 2N_c \int \frac{d\mathbf{p}}{(2\pi)^3} [n(p) - \bar{n}(p)], \quad (9)$$

where  $(Q_4 = - \int d\mathbf{x} \bar{q} i \gamma_4 q)$

$$Q_4 = \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{-ip_4}{|p_4|} [A^+(p) A(p) + B(p) B^+(p)],$$

for the diagonal component (of our interest here) and the fixed mean entropy

$$\begin{aligned}\bar{S} &= -\text{Tr}\{\rho \ln \rho\} = \\ &= -V2N_c \int \frac{d\mathbf{p}}{(2\pi)^3} [n(p) \ln n(p) + (1 - n(p)) \ln(1 - n(p)) + \bar{n}(p) \ln \bar{n}(p) + (1 - \bar{n}(p)) \ln(1 - \bar{n}(p))],\end{aligned}\quad (10)$$

( $S = -\ln \rho$ ), the minimal value of mean energy of quark ensemble

$$E = \text{Tr}\{\rho H\}.$$

The definition of mean charge (9) is given here up to the unessential (infinite) constant coming from permut-

ing the operators  $BB^+$  in the charge operator  $Q_4$ . It may not be out of place to remind that the mean charge should be treated in some statistical sense because it characterizes quark ensemble density and has no colour indices.

The contribution of free part of Hamiltonian

$$\begin{aligned}H_0 &= - \int d\mathbf{x} \bar{q}(\mathbf{x}) (i\gamma\nabla + im) q(\mathbf{x}) = \\ &= \int \frac{d\mathbf{p}}{(2\pi)^3} |p_4| [\cos\theta A^+(\mathbf{p}; s)A(\mathbf{p}; s) + \sin\theta A^+(-\mathbf{p}; s)B^+(\mathbf{p}; s) + \\ &\quad + \sin\theta B(-\mathbf{p}; s)A(\mathbf{p}; s) - \cos\theta B(\mathbf{p}; s)B^+(\mathbf{p}; s)],\end{aligned}$$

into the mean energy is calculated to be as

$$\begin{aligned}\text{Tr}\{\rho \mathcal{H}_0\} &= \int \frac{d\mathbf{p}}{(2\pi)^3} |p_4| (1 - \cos\theta) + \\ &\quad + \int \frac{d\mathbf{p}}{(2\pi)^3} |p_4| \cos\theta [n(p) + \bar{n}(p)],\end{aligned}\quad (11)$$

where  $\mathcal{H}_0 = H_0/V2N_c$  is the specific energy and  $\theta = 2\varphi$ . It is obvious that a natural regularization has been done in the first term of Eq. (11) by subtracting the free Hamiltonian contribution  $H_0$  (without pairing quarks and anti-quarks). In our particular case it is quite unaffected to have the ensemble energy equal zero at the pairing angle equal zero. It explains just an appearance of unit in the term containing  $\cos\theta$ .

The interaction part of Hamiltonian,  $\bar{q}t^a\gamma_\mu q\bar{q}'t^a\gamma_\nu q'$ , provides four nontrivial contributions. The term  $\text{Tr}\{\rho BB^+B'B'^+\}$  generates the following components:  $\bar{V}_{\alpha i}(\mathbf{p}, s)t_{ij}^a\gamma_{\alpha\beta}^\mu V_{\beta j}(\mathbf{Q}, T)\bar{V}_{\gamma k}(\mathbf{Q}, T)t_{kl}^b\gamma_{\gamma\delta}^\mu V_{\delta l}(\mathbf{p}, s)$  (adding the similar term but with the changes  $Q, T \rightarrow Q', T'$  which generates another primed quark current);  $-2\bar{V}(\mathbf{Q}, T)t^a\gamma^\mu V(\mathbf{Q}', T')\bar{V}(\mathbf{Q}', T')t^b\gamma^\mu V(\mathbf{Q}, T)$ . Here (as in the following expressions) we omitted all colour and spinor indices which are completely identical to those of previous matrix element. The term  $\text{Tr}\{\rho BAA'^+B'^+\}$  generates the following nontrivial contributions:  $\bar{V}(\mathbf{p}, s)t^a\gamma^\mu U(\mathbf{q}, t)\bar{U}(\mathbf{q}, t)t^b\gamma^\mu V(\mathbf{p}, s)$   $-\bar{V}(\mathbf{p}, s)t^a\gamma^\mu U(\mathbf{P}, S)\bar{U}(\mathbf{P}, S)t^b\gamma^\mu V(\mathbf{p}, s)$

$-\bar{V}(\mathbf{Q}, T)t^a\gamma^\mu U(\mathbf{q}, t)\bar{U}(\mathbf{q}, t)t^b\gamma^\mu V(\mathbf{Q}, T)$   $+\bar{V}(\mathbf{Q}, T)t^a\gamma^\mu U(\mathbf{P}, S)\bar{U}(\mathbf{P}, S)t^b\gamma^\mu V(\mathbf{Q}, T)$ . Averaging  $\text{Tr}\{\rho AA^+A'A'^+\}$  gives the following terms:  $\bar{U}(\mathbf{P}, S)t^a\gamma^\mu U(\mathbf{p}, s)\bar{U}(\mathbf{p}, s)t^b\gamma^\mu U(\mathbf{P}, S)$  (adding the similar term but with the changes  $P, S \rightarrow P', S'$ );  $-2\bar{U}(\mathbf{P}, S)t^a\gamma^\mu U(\mathbf{P}', S')\bar{U}(\mathbf{P}', S')t^b\gamma^\mu V(\mathbf{P}, S)$ . And eventually the nontrivial contribution comes from averaging  $\text{Tr}\{\rho A^+B^+B'A'\}$  and it has the form  $\bar{V}(\mathbf{Q}, T)t^a\gamma^\mu U(\mathbf{P}, S)\bar{U}(\mathbf{P}, S)t^b\gamma^\mu V(\mathbf{Q}, T)$ . All other diagonal matrix elements generated by the following terms  $\text{Tr}\{\rho AA^+B'B'^+\}$ ,  $\text{Tr}\{\rho BB'^+A'^+A'\}$ , do not contribute at all (their contributions equal to zero). Similar to the calculation of matrix elements performed at zero temperature in Ref. [4] we should carry out the integration over the Fermi sphere with the corresponding distribution functions in the quark and anti-quark momenta  $\int^{P_F} \frac{d\mathbf{p}}{(2\pi)^3} \rightarrow \int \frac{d\mathbf{p}}{(2\pi)^3} [n(p) + \bar{n}(p)]$  in our case of finite temperature. All necessary formulae for polarization matrices which contain the traces of corresponding spinors could be found in Refs. [3] and [4]. Bearing in mind this fact here we present immediately the result for mean energy density per one quark degree of freedom as

$$w = \frac{\mathcal{E}}{2N_c}, \quad \mathcal{E} = E/V,$$

where  $E$  is a total energy of ensemble,

$$\begin{aligned}
w = & \int \frac{d\mathbf{p}}{(2\pi)^3} |p_4| \cos \theta [n(p) + \bar{n}(p)] + 2G \int \frac{d\mathbf{p}}{(2\pi)^3} \sin(\theta - \theta_m) [n(p) + \bar{n}(p)] \int \frac{d\mathbf{q}}{(2\pi)^3} \sin(\theta' - \theta'_m) I - \\
& - G \int \frac{d\mathbf{p}}{(2\pi)^3} \sin(\theta - \theta_m) [n(p) + \bar{n}(p)] \int \frac{d\mathbf{q}}{(2\pi)^3} \sin(\theta' - \theta'_m) [n(q) + \bar{n}(q)] I + \\
& + \int \frac{d\mathbf{p}}{(2\pi)^3} |p_4| (1 - \cos \theta) - G \int \frac{d\mathbf{p}}{(2\pi)^3} \sin(\theta - \theta_m) \int \frac{d\mathbf{q}}{(2\pi)^3} \sin(\theta' - \theta'_m) I ,
\end{aligned} \tag{12}$$

(up to the constant unessential for our consideration here). In this formula the following denotes are used  $p = |\mathbf{p}|$ ,  $q = |\mathbf{q}|$ ,  $\theta' = \theta(q)$ ,  $I = I(\mathbf{p} + \mathbf{q})$  and the angle  $\theta_m(p)$  is determined by the condition as follows

$$\sin \theta_m = \frac{m}{|p_4|} .$$

It is interesting to notice that the existence of such an angle stipulates the discontinuity of mean energy functional mentioned above and found out in [3]. It is quite practical to single out the color factor in the four-fermion coupling constant as  $G = \frac{2\tilde{G}}{N_c}$ . Now let us make the following transformations while integrating in the interaction terms

$$\begin{aligned}
& 2 \int d\mathbf{p} f(p) \int d\mathbf{q} - \int d\mathbf{p} f(p) \int d\mathbf{q} f(q) - \int d\mathbf{p} \int d\mathbf{q} = \\
& = \int d\mathbf{p} f(p) \int d\mathbf{q} (1 - f(q)) - \int d\mathbf{p} (1 - f(p)) \int d\mathbf{q} ,
\end{aligned}$$

and changing the variables  $\mathbf{p} \leftrightarrow \mathbf{q}$  in the last term we obtain

$$\begin{aligned}
& \int d\mathbf{p} f(p) \int d\mathbf{q} (1 - f(q)) - \int d\mathbf{p} \int d\mathbf{q} (1 - f(q)) = \\
& - \int d\mathbf{p} (1 - f(p)) \int d\mathbf{q} (1 - f(q)) .
\end{aligned}$$

Finally we find out for the mean partial energy it looks like

$$\begin{aligned}
w = & \int \frac{d\mathbf{p}}{(2\pi)^3} |p_4| + \int \frac{d\mathbf{p}}{(2\pi)^3} |p_4| \cos \theta [n(p) + \bar{n}(p) - 1] - \\
& - G \int \frac{d\mathbf{p}}{(2\pi)^3} \sin(\theta - \theta_m) [n(p) + \bar{n}(p) - 1] \int \frac{d\mathbf{q}}{(2\pi)^3} \sin(\theta' - \theta'_m) [n(q) + \bar{n}(q) - 1] I .
\end{aligned} \tag{13}$$

We are interested in minimizing the following functional

$$\Omega = E - \mu \bar{Q}_4 - T \bar{S} , \tag{14}$$

where  $\mu$  and  $T$  are the Lagrange factors for the chemical potential and temperature respectively. The approximating Hamiltonian which we discussed above  $\hat{H}_{\text{app}}$ , is

constructed just by using the information on  $E - \mu \bar{Q}_4$  of presented functional (see, also below). For the specific contribution per one quark degree of freedom

$$f = \frac{F}{2N_c} , \quad F = \Omega/V ,$$

we receive

$$\begin{aligned}
f = & \int \frac{d\mathbf{p}}{(2\pi)^3} [|p_4| \cos \theta (n + \bar{n} - 1) - \mu (n - \bar{n})] + \int \frac{d\mathbf{p}}{(2\pi)^3} |p_4| - \\
& - G \int \frac{d\mathbf{p}}{(2\pi)^3} \sin(\theta - \theta_m) (n + \bar{n} - 1) \int \frac{d\mathbf{q}}{(2\pi)^3} \sin(\theta' - \theta'_m) (n' + \bar{n}' - 1) I + \\
& + T \int \frac{d\mathbf{p}}{(2\pi)^3} [n \ln n + (1 - n) \ln(1 - n) + \bar{n} \ln \bar{n} + (1 - \bar{n}) \ln(1 - \bar{n})] .
\end{aligned} \tag{15}$$

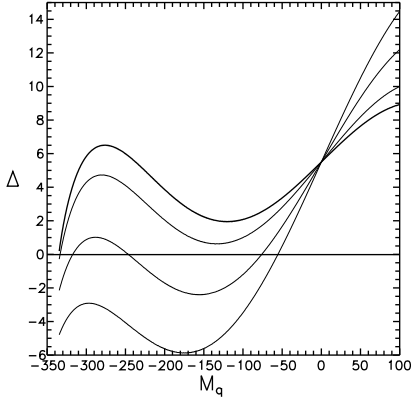


FIG. 1: The residual  $\Delta$  for equation (17) as a function of dynamical quark mass  $M_q$  (MeV) at zero value of temperature and the following values of chemical potential  $\mu$  (MeV) — 335 (the lowest curve), 340, 350, 360 (the top curve).

Here the primed variables correspond to the momentum  $q$ . The optimal values of parameters are determined by solving the following equation system ( $df/d\theta = 0$ ,  $df/dn = 0$ ,  $df/d\bar{n} = 0$ )

$$\begin{aligned} |p_4| \sin \theta - M \cos(\theta - \theta_m) &= 0, \\ |p_4| \cos \theta - \mu + M \sin(\theta - \theta_m) - T \ln\left(\frac{1}{n} - 1\right) &= 0, \\ |p_4| \cos \theta + \mu + M \sin(\theta - \theta_m) - T \ln\left(\frac{1}{\bar{n}} - 1\right) &= 0, \end{aligned} \quad (16)$$

where we denoted the induced quark mass as

$$M(\mathbf{p}) = -2G \int \frac{d\mathbf{q}}{(2\pi)^3} (n' + \bar{n}' - 1) \sin(\theta' - \theta'_m) I(\mathbf{p} + \mathbf{q}). \quad (17)$$

Turning to the presentation of obtained results in the form customary for mean field approximation we introduce a dynamical quark mass  $M_q$  parameterized as

$$\sin(\theta - \theta_m) = \frac{M_q}{|P_4|}, \quad |P_4| = (\mathbf{p}^2 + M_q(\mathbf{p}))^{1/2}, \quad (18)$$

and ascertain the interrelation between induced and dynamical quark masses. From the first equation of system (16) we fix the pairing angle

$$\sin \theta = \frac{p M}{|p_4| |P_4|},$$

and making use the identity

$$(|p_4|^2 - M m)^2 + M^2 p^2 = [p^2 + (M - m)^2] |p_4|^2, \quad (19)$$

find out that

$$\cos \theta = \pm \frac{|p_4|^2 - m M}{|p_4| |P_4|}.$$

For the clarity we choose the upper sign 'plus'. Then, as an analysis of the NJL model teaches, the branch of

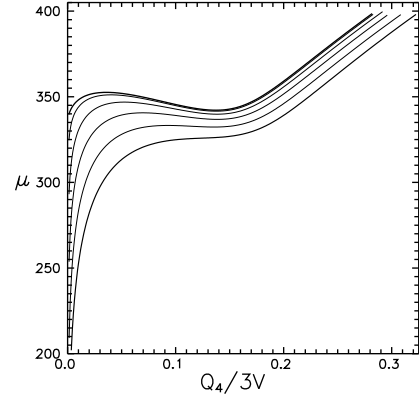


FIG. 2: The chemical potential  $\mu$  (MeV) as a function of charge density  $Q_4 = \frac{Q_4}{3V}$  (in the units of charge/fm<sup>3</sup>). The factor 3 relates the densities of quark and baryon matter. The top curve corresponds to the situation of zero temperature. The curves following down correspond to the temperature values  $T = 10$  MeV, ...,  $T = 50$  MeV with spacing  $T = 10$  MeV.

equation solution for negative dynamical quark mass is the most stable one. Let us remember here we are dealing with the Euclidean metrics (though it is not a principal point) and a quark mass appears in the corresponding expressions as an imaginary quantity. Now substituting the calculated expressions for the pairing angle into the trigonometrical factor expression (obtained in Ref. [3])

$$\sin(\theta - \theta_m) = \sin \theta \frac{p}{|p_4|} - \cos \theta \frac{m}{|p_4|}.$$

and performing some algebraic transformations of both parts of equation we come to the determination

$$M_q(\mathbf{p}) = M(\mathbf{p}) - m. \quad (20)$$

And, in particular, the equation for dynamical quark mass (17) is getting the form characteristic for the mean field approximation

$$M = -2G \int \frac{d\mathbf{q}}{(2\pi)^3} (n' + \bar{n}' - 1) \frac{M'_q}{|P'_4|} I(\mathbf{p} + \mathbf{q}). \quad (21)$$

The second and third equations of system (16) allow us to find for the equilibrium densities of quarks and anti-quarks as

$$n = \frac{1}{e^{\beta(|P_4| - \mu)} + 1}, \quad \bar{n} = \frac{1}{e^{\beta(|P_4| + \mu)} + 1}, \quad (22)$$

and, hence, the thermodynamical properties of our system as well, in particular, the pressure of quark ensemble

$$P = -\frac{dE}{dV}.$$

By definition we should calculate this derivative at constant mean entropy,  $d\bar{S}/dV = 0$ . This condition allows us, for example, to calculate the derivative  $d\mu/dV$ .

However, this way is not reliable because then the mean charge  $\bar{Q}_4$  might change, and it is more practical to introduce two independent chemical potentials — for quarks  $\mu$  and for anti-quarks  $\bar{\mu}$  (following Eq. (22) with an opposite sign). In fact, it is the only possibility to obey both conditions simultaneously. It leads to the following definitions of corresponding densities

$$n = \frac{1}{e^{\beta(|P_4| - \mu)} + 1}, \quad \bar{n} = \frac{1}{e^{\beta(|P_4| + \bar{\mu})} + 1}.$$

$$\frac{dw}{dV} = \int \frac{d\mathbf{p}}{(2\pi)^3} \left( \frac{dn}{d\mu} \frac{d\mu}{dV} + \frac{d\bar{n}}{d\bar{\mu}} \frac{d\bar{\mu}}{dV} \right) \left[ |p_4| \cos \theta - 2G \sin(\theta - \theta_m) \int \frac{d\mathbf{q}}{(2\pi)^3} \sin(\theta' - \theta'_m) (n' + \bar{n}' - 1) I \right].$$

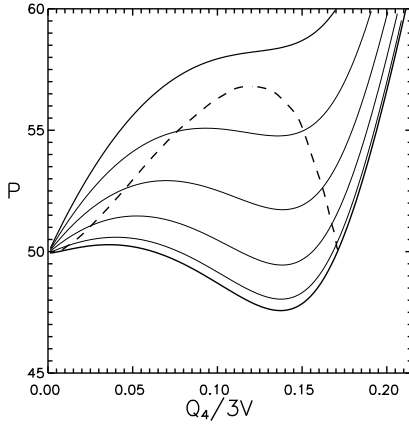


FIG. 3: The ensemble pressure  $P$  (MeV/fm<sup>3</sup>) as a function of charge density  $Q_4$  at temperatures  $T = 0$  MeV, ...,  $T = 50$  MeV with spacing  $T = 10$  MeV. The lowest curve corresponds to zero temperature. The dashed curve shows the boundary of phase transition liquid-gas, see the text.

Now expressing the trigonometric factors via dynamical quark mass and exploiting Eq. (17) we have for the ensemble pressure

$$P = -\frac{E}{V} - V 2N_c \int \frac{d\mathbf{p}}{(2\pi)^3} \left( \frac{dn}{d\mu} \frac{d\mu}{dV} + \frac{d\bar{n}}{d\bar{\mu}} \frac{d\bar{\mu}}{dV} \right) |P_4|. \quad (23)$$

The requirement of mean charge conservation

$$\frac{d\bar{Q}_4}{dV} = \frac{\bar{Q}_4}{V} + V 2N_c \int \frac{d\mathbf{p}}{(2\pi)^3} \left( \frac{dn}{d\mu} \frac{d\mu}{dV} - \frac{d\bar{n}}{d\bar{\mu}} \frac{d\bar{\mu}}{dV} \right) = 0, \quad (24)$$

provides us with an equation which interrelates the derivatives  $d\mu/dV$ ,  $d\bar{\mu}/dV$ . Apparently, here the regularized expression for mean charge of quarks and anti-quarks is meant (9). Acting in a similar way with the requirement of mean entropy conservation,  $d\bar{S}/dV = 0$ ,

In fact, this kind of description makes it possible to treat even some non-equilibrium states of quark ensemble (but with losing a covariance similar to the situation which takes place in electrodynamics while one deals with electron-positron gas). But here we are interested in the particular case of  $\bar{\mu} = \mu$ . The corresponding derivative of specific energy  $\frac{dw}{dV}$  might be presented as

we receive another equation as

$$\int \frac{d\mathbf{p}}{(2\pi)^3} \frac{dn}{d\mu} \ln \frac{n}{1-n} \frac{d\mu}{dV} - \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{d\bar{n}}{d\bar{\mu}} \ln \frac{\bar{n}}{1-\bar{n}} \frac{d\bar{\mu}}{dV} = \frac{\bar{S}}{2N_c V^2}. \quad (25)$$

Substituting here  $T \ln \frac{n}{1-n} = \mu - |P_4|$  and  $T \ln \frac{\bar{n}}{1-\bar{n}} = -\bar{\mu} - |P_4|$  after simple calculations (keeping in mind that  $\bar{\mu} = \mu$ ) we have with taking into account (24) that

$$\int \frac{d\mathbf{p}}{(2\pi)^3} \left( \frac{dn}{d\mu} \frac{d\mu}{dV} + \frac{d\bar{n}}{d\bar{\mu}} \frac{d\bar{\mu}}{dV} \right) |P_4| = -\frac{\bar{S}T}{2N_c V^2} - \frac{\bar{Q}_4 \mu}{2N_c V^2}.$$

Finally it leads for the pressure to the following expression

$$P = -\frac{E}{V} + \frac{\bar{S}T}{V} + \frac{\bar{Q}_4 \mu}{V}. \quad (26)$$

(of course, the thermodynamical potential is  $\Omega = -P V$ ). At small temperatures the anti-quark contribution is negligible, and thermodynamical description can be grounded on utilizing one chemical potential  $\mu$  only. If the anti-quark contribution is getting intrinsic the thermodynamical picture becomes complicated due to the presence of chemical potential  $\bar{\mu}$  with the condition  $\bar{\mu} = \mu$  imposed. In particular, at zero temperature the anti-quark contribution is absent and we might receive

$$P = -\mathcal{E} + \mu \rho_q,$$

where  $\mu = \sqrt{P_F^2 + M_q^2(P_F)}$ ,  $P_F$  is the Fermi momentum and  $\rho_q = N/V$  is the quark ensemble density.

For clarity, we consider mainly the NJL model [1] in this paper, i.e. the correlation function behaves as the  $\delta$ -function in coordinate space. It is well known fact in order to have an intelligent result in this model one needs to use a regularization cutting the momentum integration in Eq. (15). We adjust the standard set of parameters [5] here with  $|\mathbf{p}| < \Lambda$ ,  $\Lambda = 631$  MeV,  $m = 5.5$

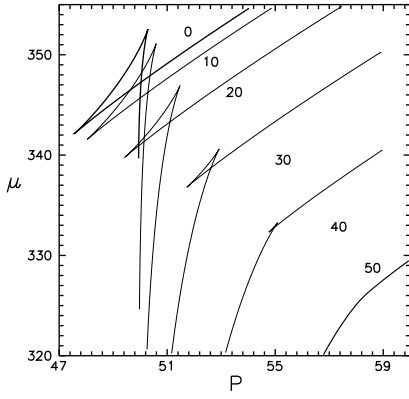


FIG. 4: The fragments of isotherms Fig. 2, 3, see text. Chemical potential  $\mu$  (MeV) as a function of pressure  $P$  MeV/fm<sup>3</sup>. The top curve corresponds to the zero isotherm and following down with spacing 10 MeV till the isotherm 50 MeV (the lowest curve).

MeV and  $G\Lambda^2/(2\pi^2) = 1.3$ . This set of parameters at  $n = 0$ ,  $\bar{n} = 0$ ,  $T = 0$  gives for the dynamical quark mass  $M_q = 335$  MeV. In particular, it may be shown the following representation of ensemble energy is valid at the extremals of functional (15)

$$E = E_{vac} + 2N_c V \int^{\Lambda} \frac{d\mathbf{p}}{(2\pi)^3} |P_4| (n + \bar{n}),$$

$$E_{vac} = 2N_c V \int^{\Lambda} \frac{d\mathbf{p}}{(2\pi)^3} (|p_4| - |P_4|) + 2N_c V \frac{M^2}{4G}, \quad (27)$$

It is easy to understand this expression with the vacuum contribution subtracted looks like the energy of a gas of relativistic particles and anti-particles with the mass  $M_q$ , and coincides identically with that calculated in the mean field approximation.

Let us summarize the results of this exercise. So, we determine the density of quark  $n$  and anti-quark  $\bar{n}$  quasi-particles at given parameters  $\mu$  and  $T$  from the second and third equations of system (16). From the first equation we receive the angle of quark and anti-quark pairing  $\theta$  as a function of dynamical quark mass  $M_q$  which is handled as a parameter. Then at small temperatures, below 50 MeV, and chemical potentials of dynamical quark mass order,  $\mu \sim M_q$ , there are several branches of solutions of the gap equation. Fig. 1 displays the difference of right and left sides of Eq. (17) which is denoted by  $\Delta$  at zero temperature and several values of chemical potential  $\mu$  (MeV) = 335 (the lowest curve), 340, 350, 360 (the top curve) as a function of parameter  $M_q$ . The zeros of function  $\Delta(M_q)$  correspond to the equilibrium values of dynamical quark mass.

The evolution of chemical potential as a function of charge density  $Q_4 = \frac{Q_4}{3V}$  (in the units of charge/fm<sup>3</sup>) with the temperature increasing is depicted in Fig. 2 (factor 3 relates the quark and baryon matter densities). The top curve corresponds to the zero temperature. The other curves following down have been calculated for the

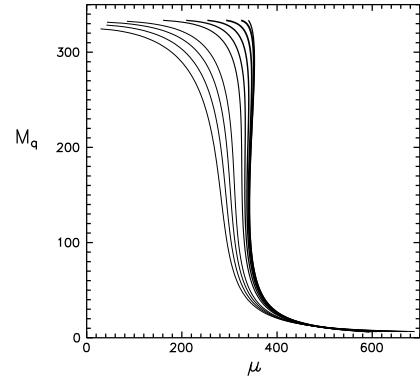


FIG. 5: The dynamical quark mass  $|M_q|$  (MeV) as a function of chemical potential  $\mu$  (MeV) at the temperatures  $T = 0$  MeV, ...,  $T = 100$  MeV with spacing  $T = 10$  MeV. The most right curve corresponds to zero temperature.

temperatures  $T = 10$  MeV, ...,  $T = 50$  MeV with spacing  $T = 10$  MeV. As it was found in Ref. [4] the chemical potential at zero temperature is increasing first with the charge density increasing, reaches its maximal value, then decreases and at the densities of order of normal nuclear matter density[9],  $\rho_q \sim 0.16/\text{fm}^3$ , becomes almost equal its vacuum value. Such a behaviour of chemical potential results from the fast decrease of dynamical quark mass with the Fermi momentum increasing. It is clear from this Fig. the charge density is still multivalued function of chemical potential at the temperature slightly below 50 MeV. The Fig. 3 shows the ensemble pressure  $P$  (MeV/fm<sup>3</sup>) as the function of charge density  $Q_4$  at several temperatures. The lowest curve corresponds to the zero temperature. The other curve following up correspond to the temperatures  $T = 10$  MeV, ...,  $T = 50$  MeV (the top curve) with spacing  $T = 10$  MeV. It is interesting to remember now that in Ref. [4] the vacuum pressure estimate for the NJL model was received as 40–50 MeV/fm<sup>3</sup> which is entirely compatible with the results of conventional bag model. Besides, some hints at instability presence (rooted in the anomalous behavior of pressure  $dP/dn < 0$ ) in an interval of the Fermi momenta has been found out.

Fig. 4 shows the fragments of isotherms of Fig. 2, 3 but in the different coordinates (chemical potential — ensemble pressure). The top curve is calculated at the zero temperature, the other isotherms following down correspond to the temperatures increasing with spacing 10 MeV. The lowest curve is calculated at the temperature 50 MeV. This Fig. obviously demonstrate a presence of the states on isotherm which are thermodynamically equilibrated and have equal pressure and chemical potential (see the characteristic Van der Waals triangle with the crossing curves). The calculated equilibrium points are shown in Fig. 3 by the dashed curve. The intersection points of dashed curve with an isotherm are fixing the boundary of gas — liquid phase transition. The corresponding straight line  $P = \text{const}$  which obeys the Maxwell rule

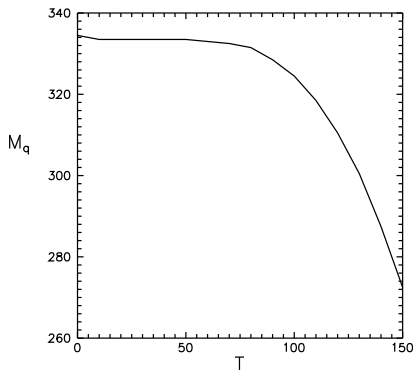


FIG. 6: The dynamical quark mass  $|M_q|$  (MeV) as a function of temperature at small value of charge density  $\mathcal{Q}_4$ .

separates the non-equilibrium and unstable fragments of isotherm and describes a mixed phase. The corresponding critical temperature for the parameter we are using in this paper turns out to be  $T_c \sim 45.7$  MeV with the critical charge density as  $\bar{\mathcal{Q}}_4 \sim 0.12$  charge/fm<sup>3</sup>. Usually the thermodynamic description is grounded on the mean energy functional which is the homogeneous function of particle number like  $E = N f(S/N, V/N)$  (without vacuum contribution). It is clear such a description requires the corresponding subtractions to be introduced, however, this operation does not change the final results considerably. It was argued in Refs. [4] that the states filled up with quarks and separated from the instability region look like 'natural construction material' to form the baryons and to understand the existing fact of equilibrium between vacuum and octet of stable (in strong interaction) baryons[10].

The dynamical quark mass  $|M_q|$  (MeV) as a function of chemical potential  $\mu$  (MeV) is presented for the temperatures  $T = 0$  MeV, ... ,  $T = 100$  MeV with spacing  $T = 10$  MeV in Fig. 5. The most right-hand curve corresponds to the zero temperature. At small temperatures, below 50 MeV, the dynamical quark mass is the multi-valued function of chemical potential. The Fig. 6 shows the dynamical quark mass as a function of temperature at small values of charge density  $\mathcal{Q}_4 \sim 0$ . Such a behaviour allows us to conclude that the quasi-particle size is getting larger with temperature increasing. It becomes clear if we remember that the momentum corresponding the maximal attraction between quark and anti-quark  $p_\theta$  (according to Ref. [3]) is defined by  $d \sin \theta / dp = 0$ . In particular, this parameter in the NJL model equals to

$$p_\theta = (|M_q| m)^{1/2}. \quad (28)$$

but its inverse magnitude defines the characteristic (ef-

fective) size of quasi-particle  $r_\theta = p_\theta^{-1}$ .

If one is going to define the quark chemical potential as an energy necessary to add (to remove) one quasi-particle (as it was shown in [4] at zero temperature),  $\mu = dE/dN$ , then in vacuum (i.e. at quark density  $\rho_q$  going to zero) quark chemical potential magnitude coincides with the quark dynamical mass. It results in the phase diagram displayed at this value of chemical potential although, in principle, this value could be smaller than dynamical quark mass as it has been considered in the pioneering paper [6]. If one takes, for example, chemical potential value equal to zero it leads to the conventional picture but, obviously, such a configuration does not correspond to the real process of filling up the Fermi sphere with quarks.

Apparently, our study of the quark ensemble thermodynamics produces quite reasonable arguments to propound the hypothesis that the phase transition of chiral symmetry (partial) restoration has already realized as the mixed phase of physical vacuum and baryonic matter[11]. However, it is clear our quantitative estimates should not be taken as ones to be compared with, for example, the critical temperature of nuclear matter which has been experimentally measured and is equal to 15 – 20 MeV. Besides, the gas component (at  $T = 0$ ) has non-zero density (as 0.01 of the normal nuclear density) but in reality this branch should correspond to the physical vacuum, i.e. zero baryonic density[12]. In principle, an idea of global equilibrium of gas and liquid phases makes it possible to formulate the adequate boundary conditions at describing the transitional layer arising between the vacuum and filled state and to calculate the surface tension effects. We are planning to consider these aspects of phase transition in a separate paper.

As a conclusion we would like to emphasize that in the present paper we demonstrated how a phase transition of liquid-gas kind (with the reasonable values of parameters) emerges in the NJL-type models in which the quarks are considered as the quasi-particles of the model Hamiltonian with four-fermion interaction. The constructed quark ensemble displays some interesting features for the nuclear ground state (for example, an existence of the state degenerate with the vacuum one) but needs further study of its role in the context of existing research [8] activity to explore the complicated (or, may be, more realistic) versions of the NJL model and knowledge of the QCD thermodynamics as gained in the lattice simulations.

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- [11] Indirect confirmation of this hypothesis one could see, for example, in the existing degeneracy of excited baryon states Ref. [7].
- [12] Similar uncertainty is present in the other predictions of chiral symmetry restoration scenarios, for example, it stretches from 2 to 6 units of normal nuclear density.